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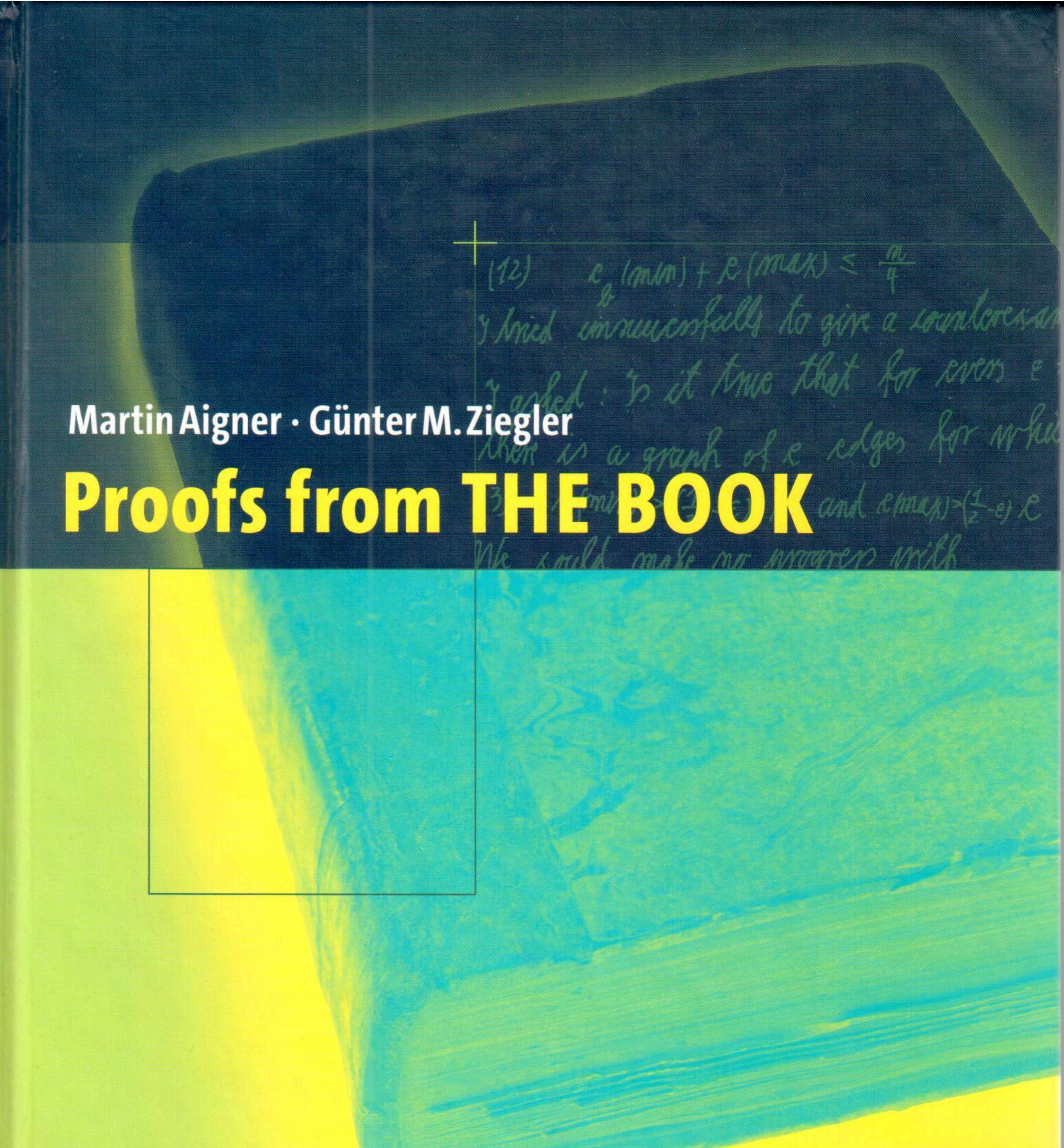
Proofs from THE BOOK



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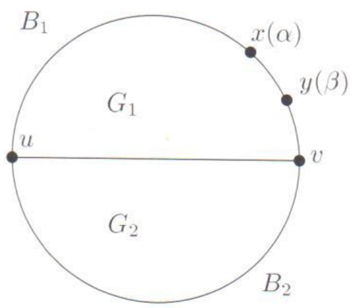
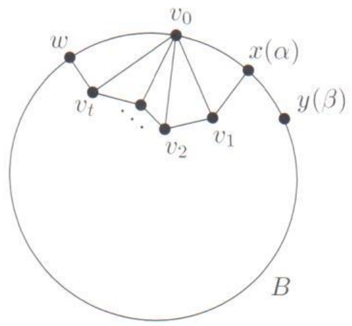
For $|V| = 3$ this is obvious, since for the only uncolored vertex v we have $|C(v)| \geq 3$, so there is a color available. Now we proceed by induction.



Case 1: Suppose B has a chord, that is, an edge not in B that joins two vertices $u, v \in B$. The subgraph G_1 which is bounded by $B_1 \cup \{uv\}$ and contains x, y, u and v is near-triangulated and therefore has a 5-list coloring by induction. Suppose in this coloring the vertices u and v receive the colors γ and δ. Now we look at the bottom part G_2 bounded by B_2 and uv. Regarding u, v as pre-colored, we see that the induction hypotheses are also satisfied for G_2. Hence G_2 can be 5-list colored with the available colors, and thus the same is true for G.



Case 2: Suppose B has no chord. Let v_0 be the vertex on the other side of the α-colored vertex x on B, and let $x, v_1, \ldots, v_t, w$ be the neighbors of v_0. Since G is near-triangulated we have the situation shown in the figure.

Construct the near-triangulated graph $G' = G \backslash v_0$ by deleting from G the vertex v_0 and all edges emanating from v_0. G' has as outer boundary $B' = (B \backslash v_0) \cup \{v_1, \ldots, v_t\}$. Since $|C(v_0)| \geq 3$ by assumption (2) there exist two colors γ, δ in $C(v_0)$ different from α. Now we replace every color set $C(v_i)$ by $C(v_i) \backslash \{\gamma, \delta\}$, keeping the original color sets for all other vertices in G'. Then G' clearly satisfies all assumptions and is thus 5-list colorable by induction. Choosing γ or δ for v_0 we can extend the list coloring of G' to all of G, and the proof is complete. □

So, the 5-list color theorem is proved, but the story is not quite over. A stronger conjecture claimed that the list-chromatic number of a plane graph G is at most 1 more than the ordinary chromatic number:

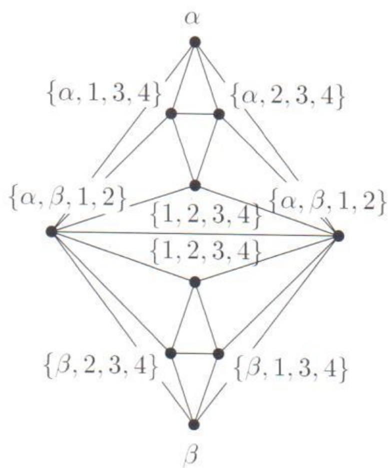
Is $\chi_\ell(G) \leq \chi(G) + 1$ *for every plane graph G ?*

Since $\chi(G) \leq 4$ by the four-color theorem, we have three cases:

Case I: $\chi(G) = 2 \implies \chi_\ell(G) \leq 3$

Case II: $\chi(G) = 3 \implies \chi_\ell(G) \leq 4$

Case III: $\chi(G) = 4 \implies \chi_\ell(G) \leq 5.$



Thomassen's result settles Case III, and Case I was proved by an ingenious (and much more sophisticated) argument by Alon and Tarsi. Furthermore, there are plane graphs G with $\chi(G) = 2$ and $\chi_\ell(G) = 3$, for example the graph $K_{2,4}$ that we considered in the preceding chapter on the Dinitz problem.

But what about Case II? Here the conjecture fails, as was first shown by Margit Voigt on a graph that was earlier constructed by Shai Gutner. His graph on 130 vertices can be obtained as follows. First we look at the "double octahedron" (see the figure), which is clearly 3-colorable. Let $\alpha \in \{5, 6, 7, 8\}$ and $\beta \in \{9, 10, 11, 12\}$, and consider the lists that are given in the figure. You are invited to check that with these lists a coloring is not possible. Now take 16 copies of this graph, and identify all top vertices and all bottom vertices. This yields a graph on $16 \cdot 8 + 2 = 130$ vertices which is still plane and 3-colorable. We assign $\{5, 6, 7, 8\}$ to the top vertex and $\{9, 10, 11, 12\}$ to the bottom vertex, with the inner lists corresponding to all 16 pairs (α, β), $\alpha \in \{5, 6, 7, 8\}$, $\beta \in \{9, 10, 11, 12\}$. For every choice of α and β we thus obtain a subgraph as in the figure, and so a coloring of the big graph is not possible.

By modifying another one of Gutner's examples, Voigt and Wirth came up with an even smaller plane graph with 75 vertices and $\chi = 3$, $\chi_\ell = 5$, which in addition uses only the minimal number of 5 colors in the combined lists. The current record is 63 vertices.

References

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